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Entanglement persistence in contact with the environment: exact results

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Abstract
The evolution of quantum entanglement of a pair of non-interacting qubits is studied for the case where one of them is non-dissipatively but dephasingly coupled to the environment. The reduced non-Markovian dynamics of the qubits is exact for an arbitrary strength of coupling to the environment and the arbitrary frequency spectrum of environment fluctuations. While for the subohmic and ohmic environments the entanglement diminishes, for the superohmic zero-temperature environment it survives for a long time.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Is it realizable to maintain the quantum entanglement of a bipartite system in a fluctuating environment as long as necessary? In ideal closed systems, the initial entanglement is preserved forever. This is not the case for open systems. Usually, interaction with the environment leads to the decay of entanglement. However, as will be shown in this communication, there are systems for which one can positively answer the above question.

A variety of aspects of entanglement in open quantum systems have attracted considerable attention due to its significance for the fundamentals and applications of quantum information processing [1]. There are several proposals how to overcome the problem of decoherence [2] by limiting quantum evolution to the decoherence free subspaces in order to maintain unitarity resulting in effectively noiseless dynamics. The way how a system interacts with its surrounding has a crucial impact on the entanglement of its components. The role of the environment can be either constructive [3, 4] or destructive, resulting in noise-induced entanglement decay and death [5] or, according to [6], the 'decoherence of entanglement'. In this context, persistent entanglement has been predicted in [7] and observed in quantum optical systems [8]. The dynamical theory of entanglement is developed under various and often quite abstract [9] assumptions. For an open system, the choice of a model of its reduced dynamics is crucial. One of the most 'popular' guidelines for that choice is the complete positivity provided by the Kossakowski–Lindblad form of the applied master equations [10]. The approach to entanglement dynamics is particularly convenient in the widely used Markovian approximation, either formal [4] or rigorously derived via Davies weak coupling theory [11]. Unfortunately, the results obtained in the weak coupling Markovian regime cannot be extrapolated to the low temperature regime. As a result, the applicability of the weak-coupling approximation for solid-state devices, often operating at deep cold, is problematic.

The key problem concerns the entanglement dynamics in the presence of a real environment derived consistently from the microscopic first principles. The dissipation and/or pure dephasing caused by such an environment is, in general, neither Markovian nor weak. Here we study such a system and show that entanglement can live for a long time. Upon the results of numerical calculation we formulate a conjecture: entanglement is persistent for systems dephasingly coupled to a superohmic bath at vanishing temperature. Even if finite, the lifetime of quantum entanglement is certainly much longer than the entanglement lifetime of the system coupled to the corresponding Markovian environment of the same strength. It is also shown that this effect is independent of a particular form of the initially entangled state. What really matters is the degree of initial entanglement.

2. Model

We study an open system consisting of two qubits, S_1 and S_2 (two-level systems, spin-1/2 particles, etc). We consider the case where only one of the qubits, say S_2, is coupled to the fluctuating environment. However, there is no exchange energy with the environment (no energy dissipation). It is pure dephasing which is an irreversible process of information loss [12]. The qubits do not interact with each other. The only connection between the qubits is information, i.e. the initial bipartite state. Different aspects of a similar system are discussed in [5]. We assume that the Hamiltonian of the system takes the form

$$H = S_1^z + S_2^z + \sum_{k=1}^{\infty} g_k \left(a_k^\dagger + a_k\right) S_2^z + \sum_{k=1}^{\infty} \omega_k a_k^\dagger a_k, \tag{1}$$

where the qubits are represented by the spin-1/2 operators S_1^z and S_2^z, the environment is modeled by harmonic oscillators, a_k and a_k^+ are the annihilation and creation Bose operators, g_k is the strength of coupling to the kth mode of the environment. Such a model may serve as a component of a simple quantum register [12].

The reduced dynamics of qubits can be determined *exactly for arbitrary model parameters* [13] provided the initial state of the total system $\varrho(0)$ can be factorized into the two-qubits state $\rho(0)$ and the state of the harmonic oscillators environment ρ_{env}, namely $\varrho(0) = \rho(0) \otimes \rho_{\mathrm{env}}$. The simplicity of the model allows for an exact, *rigorous*, treatment of entanglement dynamics beyond weak coupling and at vanishing temperature.

We assume that the environment is in an equilibrium Gibbs state ρ_{env} of temperature T and $\rho(0)$ is an arbitrary density matrix for the bipartite system. As the qubits do not interact with each other, their reduced (with respect to the environment [10, 14]) dynamics is governed by the following equation:

$$\dot{\rho}(t) = [L_1(t) + L_2(t)]\rho(t). \tag{2}$$

The dynamics of the subsystem S_1 is unitary,

$$L_1(t)(\cdot) = -\mathrm{i}\left[S_1^z, \cdot\right]. \tag{3}$$

The exact reduced dynamics of the open subsystem S_2 is governed by the generator [13]

$$L_2(t)(\cdot) = -\mathrm{i}\left[S_2^z, \cdot\right] - K(t)\left[S_2^z, \left[S_2^z, \cdot\right]\right]. \tag{4}$$

This generator is of the Kossakowski–Lindblad form and hence complete positivity is preserved [10]. The dephasing function $K(t)$ reads [13]

$$K(t) = \int_0^\infty \mathrm{d}\omega \frac{J(\omega)}{\omega} \coth(\beta\omega/2) \sin(\omega t), \tag{5}$$

where the frequency spectrum of environment fluctuations is determined by the spectral function $J(\omega) = \sum_k g_k^2 \delta(\omega - \omega_k)$. In the thermodynamic limit, it is assumed to take the form [14, 15]

$$J(\omega) = \lambda \omega^{1+\mu} \exp(-\omega/\omega_c), \qquad \mu > -1 \tag{6}$$

with the cut-off ω_c determining the largest energy scale of the environment (it removes possible problems at high frequencies) and λ corresponds to the coupling constant of the qubit and environment. The spectral exponent μ characterizes low frequency properties of the environment and defines its various types. According to the classification proposed in [15], the environment is called subohmic for $\mu \in (-1, 0)$, ohmic for $\mu = 0$ and superohmic for $\mu \in (0, \infty)$. This classification shall be reflected in the dynamical properties of entanglement.

3. Entanglement dynamics

The state of an open system is, in general, mixed. To quantify its entanglement, several measures have been proposed [16, 17]. One of the most effective operational measures is the *negativity* $N(\rho) = \max\left(0, -\sum_i \lambda_i\right)$ [17], where λ_i are negative eigenvalues of the partially transposed density matrix of the two qubits [18]. For an entangled mixed state, the negativity is positive whereas it vanishes for unentangled states. Moreover, it is an entanglement monotone and can be used to quantify the degree of entanglement.

3.1. $T = 0$

First, we analyze the limiting case $T = 0$. In this limit, the quantum mechanical properties are the most transparent because the 'classical' sources of dissipation, decoherence and dephasing, are frozen. The zero-temperature fluctuations are unavoidable due to vacuum fluctuations of the environment. It results in a non-unitary evolution with the dephasing function

$$K(t) = \lambda \frac{\Gamma(1+\mu)\omega_c^{1+\mu}}{\left(1 + \omega_c^2 t^2\right)^{(1+\mu)/2}} \sin[(1+\mu)\arctan(\omega_c t)]. \tag{7}$$

This function decays algebraically in the case of the ohmic environment, namely $K_{\mathrm{ohm}}(t) = \lambda\left(\omega_c^2 t\right)/\left(1 + \omega_c^2 t^2\right)$. Let us remind ourselves that for the model of Markovian dynamics the dephasing function does not depend on time, $K_{\mathrm{markov}}(t) = \mathrm{const.}$ [10]. However, the Markovian approximation is justified only in the regimes where the energy scale of the coupling to the environment is significantly smaller than any other energy scale in the system [10, 14]. In this sense the Markovian approximation of the real reservoir at $T = 0$ suffers serious inconsistence and can be regarded only as a formal 'toy'. The real dissipation and the Markovian toy are essentially different. In particular, $\lim_{t\to\infty} K(t) = 0$, which seems to be crucial for various effects reported below.

The characteristic feature of the time dependence of the function $K(t)$ is its non-monotonicity (see figure 1). There is a characteristic time t_c which separates two regimes: for short time, $K(t)$ increases till t_c and next it monotonically decreases. In the case of subohmic fluctuations a relatively moderate value at t_c is accompanied by a slow decay of $K(t)$ for $t > t_c$. On the other hand, in the superohmic case, the dephasing function has a sharp peak at t_c but

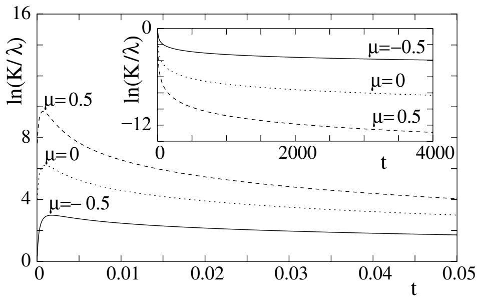

Figure 1. Short and long (inset) time evolution of the dephasing function $K(t)/\lambda$ for the subohmic, ohmic and superohmic zero-temperature environments. The cut-off frequency $\omega_c = 10^3$ has been assumed.

it rapidly vanishes for longer times. Both the sharp peak and a slow decay of $K(t)$ destroy the entanglement. However, as will be shown below, there is an 'optimal environment' with a small peak and a rapid decay, when entanglement may survive for a long time.

In the following we study the entanglement decay of the system prepared initially in the maximally entangled state,

$$\rho(0) = \tfrac{1}{2}(|01\rangle + |10\rangle)(\langle 01| + \langle 10|). \tag{8}$$

This choice is not unique. The results reported below are valid for *almost all* maximally entangled states. This conjecture has been verified numerically by choosing randomly generated maximally entangled initial states according to the algorithm elaborated in [9]. The results presented in figure 2 exhibit two qualitatively different types of behavior of entanglement dynamics. In the case of the subohmic environment, the decay to zero of entanglement is nearly exponential. For the ohmic environment, the decay is much slower. Finally, for the superohmic environment, the entanglement is long-lived, i.e. it is non-zero and tends to a constant value as time tends to infinity. However, from a practical standpoint, it is sufficient that the lifetime of entanglement is much longer than the maximal time scale in the system, here related to the first two terms on the rhs of equation (1), where the energy of the spin $\hbar\omega_0 = 1$. What follows from figure 2 is the existence of the optimal or 'best' environment for which the measure of entanglement is non-zero for long times. From our detailed analysis it follows that it corresponds to a value of the spectral exponent $\mu \approx 0.05$. Moreover, for other values of the exponent, the entanglement is still preserved in the long-time limit. Why is the superohmic environment better than the subohmic one? The reason is the power-law of the low frequency property of environment fluctuations, $J(\omega) \propto \omega^{1+\mu}$ for small ω. Because $J(\omega)$ falls off faster at low frequency in the superohmic case than in the subohmic one, the dephasing time is longer (or coherence properties are better) in systems with slower dynamics, i.e., in the superohmic environment. Moreover, as follows from [13] for a corresponding one-qubit system, only in the case of the superohmic environment, the time-independent stationary state $\lim_{t\to\infty} \rho(t)$ does not exist and the mean values of the transverse components of the spin operator relax to non-zero values.

The rate of convergence of negativity into its asymptotic value is natural evidence of entanglement survival. The other natural evidence for the system approaching its steady state is the saturation of quantum entropy $S(\rho) = -\mathrm{Tr}(\rho \ln(\rho))$. A plot of eight snapshots of both negativity and entropy versus the spectral exponent μ is presented in figure 3. The overlapping curves indicate the regime of fast convergence which confirms the previous

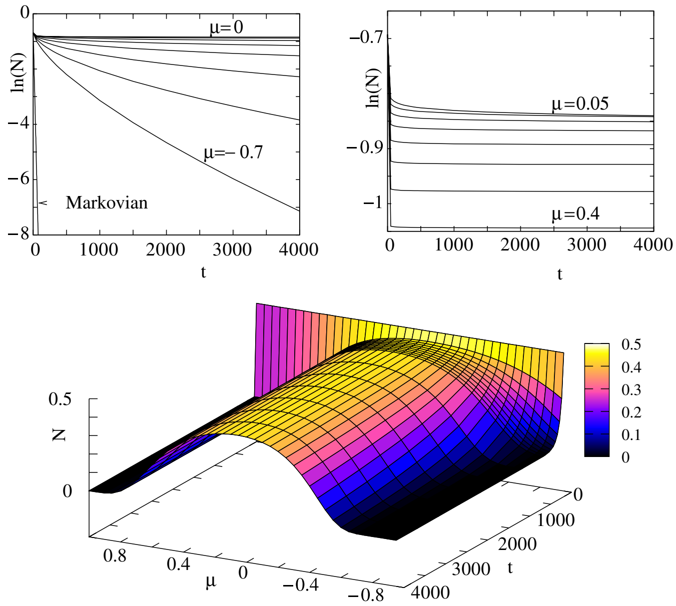

Figure 2. Dependence of the negativity N on time t and the spectral exponent μ obtained for $T = 0, \lambda = 0.01$ and $\omega_c = 10^3$. The left upper panel shows results for ohmic and subohmic environments with $\mu = 0, -0.1, \ldots, -0.7$. Here, the Markovian case is depicted as well. The right upper panel shows results for superohmic environments with $\mu = 0.05, 0.1, \ldots, 0.4$. In this case, the negativity tends to a non–zero asymptotic value, i.e., the entanglement survives despite contact with the environment.

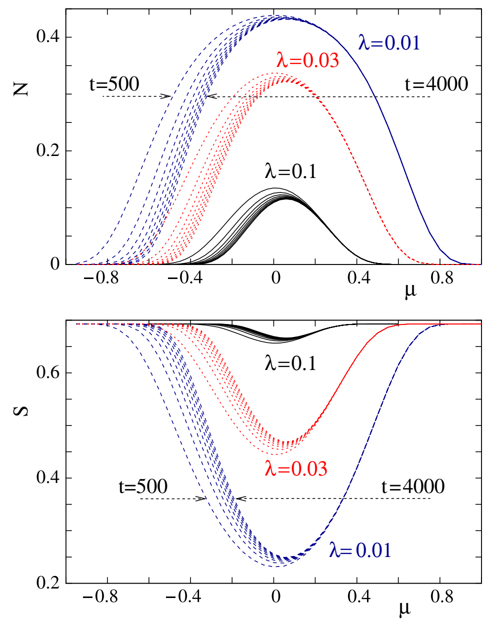

Figure 3. Snapshots of negativity (upper panel) and entropy (lower panel) for $t = 500, \ldots, 4000$ and various magnitudes of the coupling strength ($T = 0$ and $\omega_c = 10^3$).

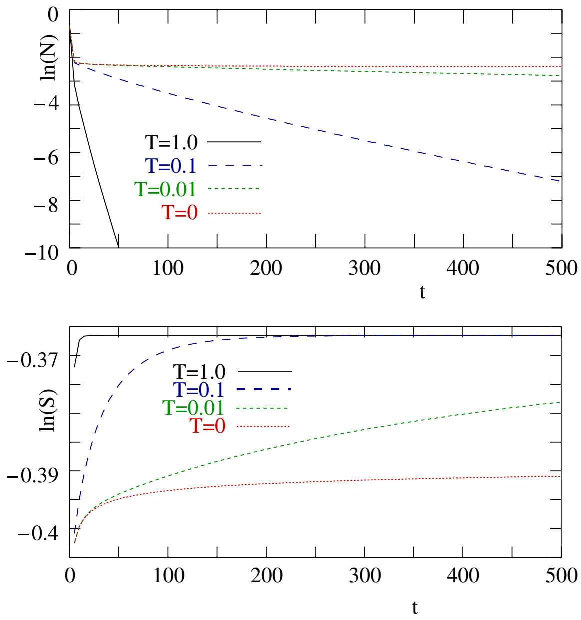

Figure 4. Time evolution of negativity (upper panel) and entropy (lower panel) at various temperatures of the superohmic bath. $\mu = 0.2$, $\lambda = 0.1$ and $\omega_c = 10^3$ have been assumed.

conclusions concerning the lifetime of the entanglement. Comparison of the results with those for the formal Markov environment shows that the Markovian approximation used beyond weak coupling strongly *overestimates* the entanglement decay.

3.2. Finite temperature and noisy initial state

Now, let us examine the case of non-zero temperature, $T \neq 0$. The intuitively natural expectation that the entanglement does not like to be warmed up is confirmed. The effect of temperature is presented in figure 4. If the temperature is non-zero, there is certainly no persistent entanglement. Fortunately, it seems that the entanglement is still long-lived for $T > 0$, i.e. its decay is slow. In particular, it is much slower than for the Markovian dynamics. Further warming up 'synchronizes' the system to the Markovian one: the weak coupling approximation starts to be applicable. The present analysis of the system at non-zero temperature suffers an inconsistence: the initial state is not affected by temperature. It is, in principle, possible to build a mixture of the state with the thermal radiation [19]. As a result the initial state becomes mixed.

Finally, we consider the noisy initial state. The preparation of a maximally entangled *pure* state is always a 'gedanken' idealization. The experimentally accessible states are always mixed due to quantum or classical noise. At $T = 0$, thermal fluctuations are frozen. The simplest model of noise is the so-called depolarizing channel [20]. It transforms the initial state into the mixed state $\rho(0) \to (p/d)1 + (1 - p)\rho(0)$, where $p \in [0, 1]$ and the dimension of the system is $d = 4$. Such a state is clearly mixed for $p \neq 0$. It is also known to remain entangled for $p < 2/3$. The numerical results, not reported here, show that the 'initial noise' affects entanglement only quantitatively: the negativity is, in the first approximation, inversely proportional to p and vanishes at $p = 2/3$.

4. Summary

In conclusion, we showed that the stability of entanglement in a system of two non-interacting qubits strongly depends on the low-frequency properties of environment fluctuations encoded in its spectral function $J(\omega)$. As a central result, it is demonstrated that for the weakly superohmic environment one may expect long-living entanglement. Our study is limited neither to Markovian nor weak coupling regimes. In our opinion, the present analysis allows for the conjecture that there are composite systems entangled forever even if coupled to a thermal bath. The results can be verified in experiments carried on, e.g., Josephson [21] and normal metal [22] flux qubits coupled to the magnetic environment in the limit of vanishing tunneling. Since one of the main sources of decoherence in such systems at low temperature is coupling to SQUID devices, it allows for an effective engineering [23] by changing the spectral function of the environment. Our findings can serve as a guideline for the optimal choice of parameters for entanglement macrobiosis. For small tunneling, the present results are a good base for the perturbative treatment of the corresponding dynamical semigroup [24] performed with respect to the tunneling term.

Acknowledgments

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